PhD Research Proposal Form
China Scholarship Council (CSC) - ENS Group

FIELD: Mathematics

Thesis subject title: **Towards multivariable overconvergence**

Name of the French doctoral school : ED 512 InfoMaths

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Lab Language : French and English

Research Proposal Abstract : Let p be a prime number and K be a finite extension of Q_p. For simplicity (to start with), we assume that K is unramified. We let f := [K : Q_p] >= 1, O_K = W(F_q) the ring of integers of K where q := p^f, and K an algebraic closure of K. We also choose an arbitrary finite extension E of Q_p containing K. We define **E** := O_E[[T]][1/T]^[1/p] where ∧ means the p-adic completion, which is a complete local field. By a celebrated theorem of Fontaine, there is an equivalence of categories ρ → D(ρ) between the category of continuous representations ρ of Gal(K^alg/K) over finite dimensional E-vector spaces and the category of (φ_q,$Z^\times_p$)-modules (also called (φ,Γ)-modules), that is finite dimensional **E**-vector spaces D(ρ) endowed with semi-linear continuous commuting actions of a Frobenius φ_q and of $Z^\times_p$ such that the image of φ_q contains a basis of D(ρ). Here O_E[[T]] is the Iwasawa algebra O_E[[Z_p]] with φ_q acting O_E-linearly by the multiplication by q on Z_p, and a ∈ $Z^\times_p$ acting O_E-linearly by the multiplication by a on Z_p.

For the purposes of p-adic analysis (among other problems), one needs a more analytic version of this equivalence. For r ∈]0, 1[⊆ R denote by C_r the rigid analytic annulus of radius [r,1[inside the rigid analytic open unit disk Spf(O_E[[T]])^{rig} and let R := lim $_{r\to 1^-}$ Γ(C_r,O_{C_r}) where the transition maps are the restrictions. This is the *Robba ring*, which also has a Frobenius φ_q and a commuting action of $Z^\times_p$. Then an important theorem of Cherbonnier and Colmez ([**CC98**]) shows that there is always a basis of D(ρ) for which both actions of φ_q and $Z^\times_p$ have matrix coefficients in R (note that the rings **E** and R do not embed one into the other). One says that D(ρ) is *overconvergent*.

The field **E** is not the only one that can be used. Let T_{LT} be a Lubin-Tate variable for the Lubin-Tate group over O_K associated to the uniformizer p. Replacing everywhere the above cyclotomic variable T by T_{LT}, one defines the field $\mathbf{E}_{LT}$ just as **E** and likewise the category of Lubin-Tate $(\varphi_q, O_K^\times)$-modules where φ_q acts on T_{LT} by the Frobenius power series and $a \in O_K^\times$ acts by its corresponding Lubin-Tate power series. Then there is again an equivalence of categories $\rho \to D_{LT}(\rho)$ between continuous representations ρ of Gal(K^alg/K) over E and Lubin-Tate $(\varphi_q, O_K^\times)$-modules $D_{LT}(\rho)$. One can also define a Lubin-Tate Robba ring R_{LT} replacing T by T_{LT}. But it turns out that, this time, only very specific $D_{LT}(\rho)$ are overconvergent, as was shown by Berger in [**Ber16**]. In fact the intuition is that, in the Lubin-Tate setting, just one variable T_{LT} is not enough for overconvergence, see [**Ber13**].

Very recently, a new kind of *multivariable* $(\varphi_q, O_K^\times)$-module in characteristic p has been defined in [**BHH⁺a**] for the purposes of the mod p Langlands program for $GL_2(K)$. The coefficient ring now has f commuting formal variables, and not just one as before. And in [**BHH⁺b**] it is proven that there is an exact and fully faithful (but not essentially surjective) functor $\rho \to D_A(\rho)$ from continuous representations of Gal(K^alg/K) over the residue field of E to these new multivariable $(\varphi_q, O_K^\times)$-modules. The construction of this functor crucially uses certain perfectoid spaces in characteristic p combined with the characteristic p variant of the above functor D_{LT}. The first part of the thesis would be to lift this construction in characteristic 0, that is to define an exact fully faithful functor $\rho \to D_A(\rho)$ from the category of continuous representations of Gal(K^alg/K) over E to a certain category of multivariable $(\varphi_q, O_K^\times)$-modules over a certain coefficient ring in characteristic 0 with f formal variables, and possibly to determine the essential image of this functor. The second part of this thesis would then be to define the Robba version of this multivariable ring in characteristic 0, and investigate if it could be true that, now, *all* the multivariable $(\varphi_q, O_K^\times)$-modules $D_A(\rho)$ are overconvergent, i.e. contain a basis such that the matrices giving φ_q and the $O_K^\times$-action have coefficients in this multivariable Robba ring.

References :

[Ber13] Laurent Berger, Multivariable Lubin-Tate (φ,Γ)-modules and filtered φ−modules, Math. Res. Lett. **20** (2013), no. 3, 409–428.

[Ber16] Laurent Berger, Multivariable (φ, Γ)-modules and locally analytic vectors, Duke Math. J. **165** (2016), no. 18, 3567–3595.

[BHH⁺a] Christophe Breuil, Florian Herzig, Yongquan Hu, Stefano Morra, and Benjamin Schraen, Conjectures and results on modular representations of $GL_n(k)$ for a p-adic field k, https://arxiv.org/pdf/2102.06188.pdf, preprint (2021).

[BHH⁺ b] Christophe Breuil, Florian Herzig, Yongquan Hu, Stefano Morra, and Benjamin Schraen, Multivariable $(\varphi, O_K^\times)$-modules and local-global compatibility, https:// arxiv.org/pdf/2211.00438.pdf, preprint (2022).

[CC98] F. Cherbonnier and P. Colmez, Overconvergent p-adic representations, Invent. Math. **133** (1998), no. 3, 581–611.

Type of PhD :

1.Full PhD

- Joint PhD/cotutelle (leading to a double diploma) : NO
- Regular PhD (leading to a single French diploma) : YES

2. Visiting PhD (for students enrolled at a Chinese institution who will be invited to a French institution to carry out a mobility period) : NO